Lecture notes

We count graphs with a labeled set of vertices, usually, $[n]$.

Ex.: There are $2^3 = 8$ distinct graphs on $[3]$, and 3 of these graphs are trees.

Here is a slight extension of the famous Cayley's Formula (proved by Borchardt in 1860).

Theorem 1 (Th. 6.1.18 in the book)**.** *For all $1 \le k \le n$, the number $b_{n,k}$ of forests of rooted trees with vertex set $[n]$ that have k components and a given set of k roots is kn^{n-k-1}. In particular, there are n^{n-2} trees with vertex set n.*

Proof. Induction on n. If $n = 1$ or $n = k$, then $b_{n,k} = 1$.

Suppose $n > k \ge 1$ and the theorem holds for all smaller $n' \ge k'$. Consider an n-vertex k-component forest F with the set K of k roots and the set R of the neighbors of these roots. By deleting K from F, we get an $(n-k)$-vertex r-component forest F' with with the set R of r roots. By definition, the number of such forests with the set of roots R is $b_{n-k,r}$. Each such F' can be extended to an n-vertex k-component forest F with the set K of roots in k^r ways. So by induction,

$$b_{n,k} = \sum_{r=1}^{n-k} \binom{n-k}{r} k^r b_{n-k,r} = \sum_{r=1}^{n-k} \binom{n-k}{r} k^r r(n-k)^{n-k-r-1}$$

$$= k \sum_{r=1}^{n-k} \binom{n-k-1}{r-1} k^{r-1} (n-k)^{n-k-1-(r-1)} = k(k+n-k)^{n-k-1}. \quad \square$$

Among ways to code a graph are adjacency and incidence matrices. For *labeled trees*, there are nicer and shorter ways to code. Consider the following procedure for a tree T with vertex set $\{1, \ldots, n\}$:

Prüfer algorithm. Let $T_0 = T$. For $i = 1, \ldots, n-1$,
(a) let b_i be the smallest leaf in T_{i-1},
(b) denote by a_i the neighbor of b_i in T_{i-1}, and
(c) let $T_i = T_{i-1} - b_i$.
The **Prüfer code** of T is the vector $(a_1, \ldots, a_{n-2})$.

EXAMPLE.

Properties of Prüfer algorithm
(P1) $a_{n-1} = n$.
(P2) Any vertex of degree s in T appears in $(a_1, \ldots, a_{n-2})$ exactly $s-1$ times.
(P3) $b_i = \min\{k : k \notin \{b_1, \ldots, b_{i-1}\} \cup \{a_i, a_{i+1}, \ldots, a_{n-2}\}\}$ for each i.

Proofs. (P1) follows from the fact that we always have a leaf distinct from n.

(P2) follows from the facts that at the moment some k appears in $(a_1, \ldots, a_{n-2})$, its degree decreases by 1 and for $s \ge 3$ the neighbors of leaves in s-vertex trees are not leaves.

(P3) follows from the algorithm and (P2). $\square$

Theorem 2 (Prüfer, 1918)**.** *Every vector $(a_1, \ldots, a_{n-2})$ with $a_i \in \{1, \ldots, n\}$ for each i is the Prüfer code of exactly one labeled n-vertex tree.*

———————————— **Here Lecture 1 ended.**

Proof. **Uniqueness**. By (P1) we know $a_{n-1} = n$. Then by (P3), we can reconstruct b_i for all $1 \leq i \leq n-1$. Thus the edges are $a_1 b_1, \ldots, a_{n-1} b_{n-1}$.

Existence. Given $(a_1, \ldots, a_{n-2})$, we let $a_{n-1} = n$ and define numbers b_i by (P3). Now consider the edges going from $a_{n-1} b_{n-1}$ backwards and check that for each i, b_i is a leaf in the graph formed by the edges $a_i b_i, \ldots, a_{n-1} b_{n-1}$. □

AN EXAMPLE.

Theorem 3 (Matrix Tree Theorem, Kirchfoff, 1847)**.** *Let G be a loopless multigraph with $V(G) = \{v_1, \ldots, v_n\}$ and $a_{i,j}$ edges connecting v_i and v_j. Let $Q = (q_{i,j})_{i,j=1}^n$, where* $q_{i,j} = \begin{cases} d(v_i), & \text{if } j = i; \\ -a_{i,j}, & \text{if } j \neq i. \end{cases}$ *Let $Q_{s,t}$ be obtained from Q by deleting row s and column t. Then $\tau(G) = (-1)^{s+t} \det Q_{s,t}$.*

Laplace extension: *Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix. Then*
1) For each i, $\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$.
2) For each $i_2 \neq i_1$, $\sum_{j=1}^n (-1)^{i_1+j} a_{i_1 j} \det A_{i_2 j} = 0$.

Lemma 4. *Let $A = (a_{ij})_{i,j=1}^n$ be a matrix with columns $A_1, \ldots, A_n$. If $\sum_{j=1}^n A_j = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$, then for each i and each j_1, j_2,*

$$(-1)^{j_1} \det A_{ij_1} = (-1)^{j_2} \det A_{ij_2}.$$

Proof in class and Lemma 6.1.24 in the book.

Lemma 5 (Binet-Cauchy Formula)**.** *Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ji})$ be an $m \times n$ matrix, $C = AB$. For $S \subset [m]$ with $|S| = n$, let A_S (respectively, B_S) denote the $n \times n$ submatrix of A (respectively, of B) formed by the columns (respectively, rows) indexed by S. Then*

$$\det A = \sum_{S \subset [m]:|S|=n} \det A_S \det B_S.$$

This is a HOMEWORK PROBLEM.

Proof of Matrix Tree Theorem. (1) Let D be any orientation of G and M be its incidence matrix. Then $Q = MM^T$.

(2) Let B be any $(n-1) \times (n-1)$-submatrix of M. Then $\det B = 0$ if the corresponding $n-1$ edges in G form a subgraph with a cycle. Otherwise, $\det B \in \{-1, 1\}$.

Let M^* be obtained from M by deleting row n. Then $Q^* = M^*(M^*)^T$.

(3) Calculate $\det Q^*$ by Lemma 5: every term is 0 or 1, and 1 if the edges in S form a tree.

———————————— **Here Lecture 2 ended.**

A *branching* or *out-tree* is an orientation of a tree that directs all edges from a given vertex (a *root*).

An *arborescence* is a digraph whose every component is a branching. An *in-tree* is a reversed branching.

For a digraph G with incidence matrix A, let D^+ (resp. D^-) be the diagonal matrix of in-degrees (resp. out-degrees), $Q^+ = D^+ - A^T$ and $Q^- = D^- - A^T$.

Examples.

Theorem 6 (Directed Matrix Tree Theorem, Tutte, 1948, Th. 6.1.28 in the book)**.** *The number of spanning out-trees (in-trees) of G rooted at v_i is the value of the cofactor for any entry in ith row of Q^- (ith column of Q^+).*

Examples.

Instead of Theorem 6, we will prove a much more general theorem:

Theorem 7 (Matrix Arborescence Theorem, Chaiken–Kleitman, 1978, Th. 6.1.30 in the book)**.** *For real a_{ij}, variables $x_1, \ldots, x_n$ and an arborescence A on $\{v_1, \ldots, v_n\}$, let $w_A = \prod_{v_j v_i \in E(A)} a_{ij} x_j$. For $S \subseteq [n]$, let $T(S)$ be the set of all arborescences on $\{v_1, \ldots, v_n\}$ whose set of roots is $\{v_i : i \in S\}$. Define $Q = (q_{ij})_{i,j=1}^n$ as follows:*

$$q_{ij} = \begin{cases} -a_{ij} x_j, & i \neq j; \\ \sum_{\ell \neq i} a_{i\ell} x_\ell, & i = j. \end{cases}$$

If Q_S is obtained from Q by deleting all rows and columns indexed by S, then

$$\det Q_S = \sum_{A \in T(S)} w_A.$$

Observation. Theorem 6 is obtained from Theorem 7 by letting a_{ij} be the number of edges from v_j to v_i, letting all $x_j = 1$ and S be a singleton.

EXAMPLES.

Proof of Theorem 7. By induction on $m = n - s$, where $s = |S|$. If $n = s$, then we get $1 = 1$. Suppose the theorem holds for $n - s \leq m - 1$. Consider any choice of $S \subset [n]$ with $|S| = s$ and any a_{ij}s. We view $\det Q_S$ as a polynomial of degree m, $f_S(x_1, \ldots, x_n)$. For $i \in S$, call x_i a *root variable.*

Two claims:

(1) In both, $\sum_{A \in T(S)} w_A$ and $f_S(x_1, \ldots, x_n)$ each term has degree 0 in some **non-root** variable

(2) For each non-root variable x_i, the terms in which x_i is missing coincide in $\sum_{A \in T(S)} w_A$ and $f_S(x_1, \ldots, x_n)$.

Together, the claims imply the theorem, so let us prove them.

———————————— **Here Lecture 3 ended.**

Proof of (1). Since $k < n$, in w_A there are non-root vertices. The outdegree of a non-root leaf v_i is 0, and hence x_i is not present.

Consider $\det Q_S$. Recall that the sum of columns of Q is the zero vector by definition. When we delete rows and columns corresponding to S, this is not true because in the diagonal elements some terms with x_j for $j \in S$ may remain. But when we set all these variables to 0, the property recovers. So $f_S \mid_{x_j=0, j\in S} \equiv 0$. This means each term of Q_S contains x_j for some $j \in S$. Since the degree of each term is m, some of the m non-root variables is missing. $\square$

Proof of (2). Consider the terms with no non-root x_t in both polynomials. In $\sum_{A\in T(S)} w_A$ they arise from the arborescences where x_t is a leaf. Each such arborescence A is obtained from an arborescence A' with $n-1$ vertices by adding an arc to v_t. So if T' is the set of all arborescences on $V(G) - v_t$, then the sum of terms omitting x_t is

$$\Big(\sum_{A' \in T'(S)} w_{A'} \Big) \Big(\sum_{j \neq t} a_{t,j} \cdot x_j \Big).$$

In f_S the terms omitting x_t form $f_S(x_1, \dots, x_{t-1}, 0, x_{t+1}, \dots, x_n)$. The only non-zero entry in the ts column of this determinant is $\sum_{j\neq t} a_{t,j} x_j$ in row t. Expand the determinant w.r.t. this column: By the IH, the remaining determinant equals $\Big(\sum_{A'\in T'(S)} w_{A'} \Big)$. $\square$

Together, the claims prove the theorem. $\square$

AN EXAMPLE.

Eulerian circuits versus trees in digraphs

Lemma 8 (Lem. 6.1.33 in the book)**.** *For each Eulerian circuit in a digraph G that begins from vertex v along edge e, the set T of edges last leaving each vertex apart from v forms an in-tree with root v.*

Proof. The outdegree in T of each vertex apart from v is 1, the outdegree of v is 0, and there are no directed cycles. $\square$

Algortithm.
Input. An Eulerian digraph D and a spanning in-tree T.
Step 1. For each $u \in V(D)$, give an order of exiting edges s.t.
(*) for each $u \neq v$, the edge of T is the last.
Step 2. Starting from v always go along the non-used edges smallest in the order.

Lemma 9 (Lem. 6.1.35 in the book)**.** *The algorithm above always produces an Eulerian circuit in D.*

Proof. We check that by (*) the our trail L can stop only at v. Hence L uses all edges entering v. Then for each in-neighbor w of v, L also uses all edges entering w. Continuing, we conclude that L uses all edges at each vertex. $\square$

Theorem 10 (BEST Theorem, de Bruijn–van Aardenne-Ehrenfest, 1951, Smith–Tutte, 1941, Th. 6.1.36 in the book)**.** *Let D be an Eulerian digraph with $V(D) = \{v_1, \dots, v_n\}$,* *where $d^+(v_i) = d^-(v_i) = d_i$ for all $1 \leq i \leq n$. Let $M = M_j$ be the number of spanning in-trees in D with root v_j. Then the number of Eulerian circuits in D is*

$$M \prod_{i=1}^{n} (d_i - 1)!.$$

Proof. For each in-tree, the algorithm produces $\prod_{i=1}^{n}(d_i - 1)!$ distinct Eulerian circuits, and by Lemma 8, each Eulerian circuit is obtained this way. □

Corollary 11. *In each Eulerian digraph, the number of spanning in-trees with root v_i is equal for all v_i (and equal to the number of spanning out-trees with root v_i).*

Corollary 12. *In each Eulerian digraph, the number of Eulerian circuits can be computed in polynomial time.*

Note that for undirected graphs it is NP-complete to calculate the number of Eulerian circuits.

———————————— **Here Lecture 4 ended.**

Decompositions and graceful labelings

A *decomposition* of a graph G is a set of edge-disjoint subgraphs of G whose union is G. An *H-decomposition* of G is a decomposition in which each subgraph is H.

Conjecture (Ringel, 1964). For each m-edge tree T, K_{2m+1} has a T-decomposition.

Examples.

Solution by Montgomery, Pokrovsky and Sudakov.

A *graceful labeling* of a graph G with m edges is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$ such that $\{|f(u) - f(v)| : uv \in E(G)\} = [m]$.

Examples.

Conjecture (Kotzig, 1964?). Every tree has a graceful labeling.

Theorem 13 (Rosa, 1967, Th. 6.1.41 in the book)**.** *If a graph T with m edges has a graceful labeling, then K_{2m+1} has a T-decomposition.*

Proof. View $0, 1, \ldots, 2m$ as vertices of a cycle C_{2m+1}. The *difference* between i and j is the distance in C_{2m+1} between them. The set E_i of the edges in K_{2m+1} has $2m + 1$ edges forming a 2-factor. When we place the vertices of T onto C_{2m+1} according to its graceful labeling, all edges of T are in different E_i. Rotating T around the cycle never uses the same edges, and each edge will be in exactly one copy of T. □

Caterpillars have graceful labelings. - The idea in the lecture and in the book.

Theorem 14 (Wilson, 1976, Th. 6.1.49 in the book)**.** *For a graph H with m edges, let $q = q(H)$ be the* **gcd** *of the vertex degrees of H. There is n_H s.t. for each $n \geq n_H$ with $m \mid \binom{n}{2}$ and $q \mid (n-1)$ graph K_n has an H-decomposition.*

Conjecture (Graham–Häggkvist). For each m-edge tree T, each $2m$-regular graph has a T-decomposition and each m-regular bipartite graph has a T-decomposition.

Theorem 15 (Th. 6.1.52 in the book). *Let T be a tree with m edges. If a $2m$-regular graph G has a 2-factorization s.t. no cycle in G of length at most $1 + \mathrm{diam}(T)$ has all its edges in different factors, then G has a T-decomposition.*

Proof. Let $F = (F_1, \ldots, F_{2m})$ be such a "good" 2-factorization. By induction on m we prove that for any injective marking f of the vertices of T, there is a T-decomposition of G s.t.
(a) each vertex of G appears in $m + 1$ copies of T **once with each name**, and
(b) each copy of T uses one edge from each F_i.

———————————— **Here Lecture 5 ended.**

For $m = 1$ the claim is immediate. Suppose it is proved for all $m' < m$. Let f be a vertex marking of T and i is a leaf in T with neighbor j. By induction there is such decomposition of $G - E(F_m)$ into copies of $T' = T - i$. Fix a cyclic orientation D of all cycles in F_m. For each $v \in V(G)$, let v' be the unique successor of v in F_m. By (a), there is a unique copy $T'(v)$ of T' in which v plays the role of j. Extend $T'(v)$ to a copy $T'(v)$ of T by adding edge vv'. The main observation is that v' is not in $V(T'(v))$, since otherwise by (b) we would get a cycle all whose edges are in distinct F_h. □

Conjecture (Gyárfás). For any trees $T_1, \ldots, T_{n-1}$ where T_i has i edges, K_n decomposes into $T_1, \ldots, T_{n-1}$.

AN EXAMPLE.

Graph packing

A *packing* of n-vertex graphs $G_1, \ldots, G_k$ is an expression of them as edge-disjoint subgraphs of K_n.

Two n-vertex graphs G and H pack if G is a subgraph of the complement of H, equivalently, if H is a subgraph of the complement of G.

1978: Sauer–Spencer, Bollobás–Eldridge, Catlin.

Theorem 16 (Sauer–Spencer, 1978, Prop. 6.1.56 in the book). *If G and H are n-vertex graphs and $|E(G)| \cdot |E(H)| < \binom{n}{2}$, then G and H pack. The restriction is sharp.*

Proof. There are $n!$ bijections from $V(G)$ to $V(H)$. For each $e \in E(G)$, $f \in E(H)$, there are exactly $2(n-2)!$ such bijections mapping e onto f. So, the total number of bijections mapping **some** edge of G onto **some** edge of H is less than

$$\binom{n}{2} \cdot (2(n-2)!) = n!.$$

Hence there is a bijection that is a packing.

Examples: 1) K_n and a K_2, 2) Star and perfect matching when n is even. □

———————————— **Here Lecture 6 ended.**

Theorem 17 (Catlin, 1974, Sauer–Spencer, 1978, Th. 6.1.57 in the book). *If G and H are n-vertex graphs and $2\Delta(G)\Delta(H) < n$, then G and H pack. The restriction is sharp.*

Proof. Let σ be a bijection from $V(G)$ to $V(H)$ with the least edges of G mapped onto edges of H. Let $V(G) = \{x_1, \ldots, x_n\}$ and $V(H) = \{y_1, \ldots, y_n\}$ so that $\sigma(x_i) = y_i$ for all i. W.l.o.g., we may assume that $x_1x_n \in E(G)$ and $y_1y_n \in E(H)$. We refer to edges of G as "red" and to edges of H as "blue". If we switch the image of x_n with the image of x_k for some $2 \leq k \leq n-1$, then there are two dangers: (a) there exists j s.t. $x_n x_j \in E(G)$ and $y_k y_j \in E(H)$ or (b) there exists i s.t. $x_k x_i \in E(G)$ and $y_n y_i \in E(H)$.

The k not good by (a) are blue neighbors of red neighbors of x_n. The total number of blue neighbors of red neighbors of x_n is at most $\Delta(G)\Delta(H)$, and one of them is x_n itself, so (a) excludes at most $\Delta(G)\Delta(H) - 1$ options for $2 \leq k \leq n-1$. Similarly, (b) also excludes at most $\Delta(G)\Delta(H) - 1$ options, at so at least one switch would decrease the number of edges of G mapped onto edges of H, contradicting the choice of σ. □

Examples of sharpness!

Conjecture, Bollobás–Eldridge-Catlin. If G and H are n-vertex graphs and $(\Delta(G)+1)(\Delta(H)+1) \leq n+1$, then G and H pack.

Examples of sharpness!

Aigner-Brandt and Alon-Fisher: $\Delta(G) \leq 2$. Csaba-Shokoufandeh-Szemerédi: $\Delta(G) = 3$ for huge n.

Equitable coloring

A proper k-coloring of a graph G is *equitable* if the sizes of all color classes differ by at most 1. In terms of packing, this is a packing of a graph with the n-vertex graph that is a disjoint union of cliques of almost the same size.

Examples: $K_{1,n-1}$, $K_{2m+1,2m+1}$.

Not monotone!

Theorem 18 (Hajnal-Szemerédi, 1970, Th. 6.1.60 in the book). *For each graph G and each $k > \Delta(G)$, G has an equitable k-coloring.*

Proof. Step 1: Prove that it is enough to consider the case $|V(G)|$ divisible by k, say $|V(G)| = n = ks$.

Step 2: Use induction on $|E(G)|$ for graphs with max degree at most $k-1$. The base is trivial. Suppose the theorem holds for all graphs with less than m edges and G has m edges and max degree at most $k-1$.

Construct a *near-equitable k-coloring* of G. Let the color classes be $V_1, \ldots, V_k$, where $|V_1| = s-1$, $|V_2| = \ldots = |V_{k-1}| = s$, and $|V_k| = s+1$.

——————————— **Here Lecture 7 ended.**

Step 3: Define a digraph H with $V(H) = \{V_1, \ldots, V_k\}$ and $V_iV_j \in E(H)$ iff some $w \in V_i$ has no neighbors in V_j. Say that V_i is accessible if H has a V_i, V_1-path.

If V_k is accessible, then we are done. So suppose not. Choose a near-equitable k-coloring f with the fewest inaccessible classes. Let $\mathcal{A}$ be the set of accessible classes and $\mathcal{B} = V(H) - \mathcal{A}$. So $V_1 \in \mathcal{A}$, $V_k \in \mathcal{B}$.

Let $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $A \bigcup_{W \in \mathcal{A}} W$, $B \bigcup_{W \in \mathcal{B}} W$.

For $W, X \in \mathcal{A}$, W *blocks* X if $H - W$ has no X, V_1-path. In particular, V_1 blocks all. If $W \in \mathcal{A}$ blocks no other $X \in \mathcal{A}$, it is called *free*. Let $\mathcal{A}'$ be the set of free classes, $a' = |\mathcal{A}|'$, $A' \bigcup_{W \in \mathcal{A}'} W$.

A vertex $x \in X \in \mathcal{A}'$ is *moveable* if there is another $W \in \mathcal{A}$ s.t. x has no neighbors in W.

For $x \in X \in \mathcal{A}'$ and $y \in B$, x is the *solo neighbor* of y if x is the unique neighbor of y in X. In this case, edge xy is a *solo edge*.

Step 4: **Claim 1.** If $x \in X \in \mathcal{A}'$ is moveable, $y \in Y \in \mathcal{B}$ and xy is a solo edge, then G has an equitable k-coloring.

Proof: in the book and in class.

Step 5: **Claim 2.** If $x \in X \in \mathcal{A}'$ is not moveable, $y, y' \in \mathcal{B}$ and xy, xy' are solo edges, then G has a near-equitable k-coloring with fewer inaccessible classes.

Proof. Each $y \in \mathcal{B}$ has at least a neighbors in A and hence $< b$ neighbors in B. So by induction $G[B - y]$ has an equitable b-coloring g.

Since x is not moveable, $d_{A+y}(x) \geq a$ and so $d_{B-y}(x) < k - a = b$. Hence we can add x to some of b classes, and call the new class V_k.

Since x was a solo neighbor of y in X, the set $X' = X - x + y$ is independent. So we get a new near-equitable coloring g' of G. Since X was free, each other class in $\mathcal{A}'$ is accessible. Since x was not moveable, some other vertex was, so X' is in the new $\mathcal{A}$. Since $y'x$ was solo, and $yy' \notin E(G)$, the class of y' is also accessible now.

Step 6: We will show now that always the conditions of Claim 1 or the conditions of Claim 2 are satisfied.

Case 1: $a' \leq b$. Let $\mathcal{A}'' = \mathcal{A} - \mathcal{A}'$. Note $V_1 \in \mathcal{A}''$. There is $W \in \mathcal{A}''$ that blocks only classes in $\mathcal{A}'$. Let $V_j \in \mathcal{A}'$ be blocked by W. Then $d_A(x) \geq a - a' - 1$ for each $x \in V_j$, so by the case,

$$d_B(x) \leq k - (a - a' - 1) \leq b + a' \leq 2b. \tag{1}$$

Let $U = \{x \in V_j : x \text{ is a solo neighbor for some } y \in B\}$ and $U' = V_j - U$.

———————————— **Here Lecture 8 ended.**

By (1), $|E(U', B)| \leq 2b|U'|$. By definition, each $y \in B$ either has a neighbor in U or at least 2 neighbors in U'. So $2(|B| - |N_B(U)|) \leq 2b|U'|$. Hence

$$bs + (a-1)|U| = b(|U'| + |U|) + (a-1)|U| = b|U'| + (k-1)|U|$$
$$\geq |B| - |N_B(U)| + |N_B(U)| + \sum_{x \in U} d_A(x) = bs + 1 + \sum_{x \in U} d_A(x).$$

This means $(a-1)|U| > \sum_{x \in U} d_A(x)$, and so some $x \in U$ is movable, thus we can apply Claim 1.

Case 2: $a' \geq b$. Let I be a maximal independent subset of B with $|I| \geq s$. For $y \in I$, let $\sigma(y)$ be the number of solo edges incident to y. Since all classes in $\mathcal{B}$ are inaccessible, $d_A(y) \geq a + a' - \sigma(y)$; so

$$\sigma(y) \geq a' + a - d_A(y) \geq a' - b + d_B(y) + 1.$$

By the maximality of I, $\sum_{y \in I}(d_B(y) + 1) \geq |B| = bs + 1$. Since $(a' - b) \geq 0$, $|I|(a' - b) \geq s(a' - b)$. Hence

$$\sum_{y \in I} \sigma(y) \geq \sum_{y \in I}(a' - b + d_B(y) + 1) \geq |I|(a' - b) + \sum_{y \in I}(d_B(y) + 1)$$
$$\geq s(a' - b) + bs + 1 > |A'|.$$

Hence some $x \in A'$ is incident to two such solo edges. If x is moveable, then we can apply Claim 1, otherwise, we can apply Claim 2. $\square$

Chen-Li-Wu Conjecture: If $k \geq 3$ and G is a connected graph with $\Delta(G) \leq k$ distinct from K_{k+1} and for $K_{k,k}$, then G is equitably k-colorable.

Section 2: Vertex degrees

A sequence $(d_1, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ is *graphic* if there is an n-vertex (simple) graph whose degree sequence is $(d_1, \ldots, d_n)$.

Theorem 19 (Erdős and Gallai, 1960, Th. 6.2.10 in the book)**.** *A non-increasing sequence* $\mathbf{d} = (d_1, \ldots, d_n)$ *of nonnegative integers is graphic iff* $d_1 + \ldots + d_n$ *is even and*

$$(2) \qquad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \qquad \forall k \in [n].$$

Proved in Math581.

Theorem 20 (Havel 1955, Hakimi 1962, Th. 6.2.5 in the book)**.** *The only graphic sequence of length* 1 *is* (0)*. For* $n > 1$ *a sequence* $\mathbf{d} = (d_1, \ldots, d_n)$ *of integers with* $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ *is graphic if and only if the sequence* $\mathbf{d}' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ *is graphic.*

——————————— **Here Lecture 9 ended.**

Example: $(5, 5, 3, 3, 2, 2, 1, 1)$.

Proof. ($\Longleftarrow$) Suppose $\mathbf{d}'$ is graphic. Let G' be a simple graph with degree sequence $\mathbf{d}'$ and vertex set $\{v_2, \ldots, v_n\}$ where $d_{G'}(v_i) = d'_{i-1}$.

Let G be the graph obtained by adding to G' a new vertex v_1 adjacent to $v_2, \ldots, v_{d_1+1}$. Then the degree sequence of G is $\mathbf{d}$. Thus $\mathbf{d}$ is graphic.

($\Longrightarrow$) Suppose $\mathbf{d}$ is graphic. Among the simple graphs with degree sequence $\mathbf{d}$ and vertex set $V = \{v_1, v_2, \ldots, v_n\}$ where the degree of v_i is d_i for all i, choose a graph G in which

$$(3) \qquad v_1 \text{ has the most neighbors in } S = \{v_2, \ldots, v_{d_1+1}\}.$$

If $N_G(v_1) = S$, then the degree sequence of $G - v_1$ is $\mathbf{d}'$, and hence $\mathbf{d}'$ is graphic. Thus assume v_1 is not adjacent to some $v_i \in S$.

In this case, v_1 has a neighbor $v_j \notin S$. Since $i < j$, $d_i \geq d_j$. Moreover, v_i is not adjacent to v_1 while v_j is. Together with $d_i \geq d_j$, this yields that there is $v_k \in V$ adjacent to v_i but not to v_j.

Then the graph G_1 obtained from G by deleting edges v_1v_j and v_iv_k and adding edges v_1v_i and v_jv_k is a simple graph with the same degree sequence as G. But in this graph, v_1 has more neighbors in S, contradicting (3). $\square$

Definition of 2-*switches.*

Theorem 21 (Many people, Th. 6.2.7 in the book)**.** *If* G *and* H *are graphs on the same vertex set* V*, then* $d_G(v) = d_H(v)$ *for all* $v \in V$ *iff one can transform* G *into* H *by a sequence of* 2*-switches.*

Proof. ($\Longleftarrow$) Evident.

($\Longrightarrow$) By induction on n. For $n \leq 3$ degree sequence defines the graph. Let $n \geq 4$. Rename the vertices in V so that $d_G(v_1) \geq d_G(v_2) \geq \ldots \geq d_G(v_n)$. Let $d_G(v_1) = D$. By the proof of Theorem 20, via 2-switches we can get from G a graph G^* s.t. $N_{G^*}(v_1) = \{v_2, \ldots, v_{D+1}\}$ and get from H a graph H^* s.t. $N_{H^*}(v_1) = \{v_2, \ldots, v_{D+1}\}$.

Let $G' = G^* - v_1$ and $H' = H^* - v_1$. The degrees of the vertices in G' and H' coincide. Thus by induction we can transform G' to H'. Since these 2-switches do not involve edges incident to v_1, these switches also transform G^* to H^*. So, we can transform by 2-switches: $G \to G^* \to H^* \to H$. □

Theorem 22 (Edmongs 1964, Th. 6.2.23 in the book)**.** *For $k \geq 2$, a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of nonnegative integers is the degree sequence of a k-edge-connected graph iff $\mathbf{d}$ is graphic and* $\min\{d_i : 1 \leq i \leq n\} \geq k$.

Proof. ($\Longrightarrow$) Evident.

($\Longleftarrow$) Find a realization of $\mathbf{d}$ with the maximum edge connectivity. Suppose it is $h < k$. Than among h-edge-connected realizations of $\mathbf{d}$ choose G with the fewest edge cuts with h edges.

Since $k \geq 2$, $h \geq 1$! (May use hw).

Choose an inclusion minimal $A \subset V(G)$ with $|E_G(A, V(G)-A)| = h$. Let B be a smallest subset of $V(G) - A$ s.t. $|E_G(B, V(G) - B)| = h$. Let P be a shortest A, B-path with $x \in A \cap P$ and $y \in B \cap P$.

Since $|E_G(A, V(G) - A)| < k$, there is a neighbor $w \in A$ of x with $N(w) \subseteq A$! Similarly, there is a neighbor $z \in B$ of y with $N(z) \subseteq B$. Construct G' by deleting xw an yz and adding xz and yw. Then $|E_{G'}(A, V(G) - A)| > h$. The theorem will be proved when we show the following claims:

(i) $\kappa'(G') \geq h$ and

(ii) *No new U with $|E_{G'}(U, V(G) - U)| = h$ appear.*

If at least one of (i) and (ii) does NOT hold, then there is $D \subset V(G)$ s.t.

(4) $$|E_{G'}(D, V(G) - D)| \leq h \text{ and } |E_{G'}(D, V(G) - D)| < |E_G(D, V(G) - D)|.$$

Then we may assume $\{x, z\} \subseteq D$ and $\{y, w\} \subseteq \overline{D}$. In particular, $A \cap D \neq \emptyset$ and $A - D \neq \emptyset$. Note

$$h \geq |E_{G'}(D, V(G) - D)| = |E_{G'}(D \cap A, A - D)| + |E_{G'}(D \cap B, B - D)| + |E_{G'}(D \cap A, \overline{A} - D)|$$
$$+ |E_{G'}(D \cap B, \overline{B} - D)| + |E_{G'}(D - A - B), \overline{D} - A - B)|.$$

Since one of the edges in P connects D with $\overline{D}$, $|E_{G'}(D \cap A, A - D)| + |E_{G'}(D \cap B, B - D)| \leq h - 1$.

———————————— **Here Lecture 10 ended.**

Assume by symmetry that $|E_{G'}(D \cap A, A - D)| \leq \lfloor (h-1)/2 \rfloor$. Then $|E_G(D \cap A, A - D)| \leq \lfloor (h-1)/2 \rfloor$.

Again by symmetry, $|E_G(D \cap A, \overline{A})| \leq \lfloor (h)/2 \rfloor$. Then

$$|E_G(D \cap A, \overline{A \cap D})| \leq \lfloor (h-1)/2 \rfloor + \lfloor h/2 \rfloor \leq h - 1, \text{ a contradiction.} \qquad \square$$

Theorem 23 (Lovász, 1966 (born 1948)), Th. 6.2.29 in the book)**.** *Let G be a graph. If $D_1, \ldots, D_t$ are nonnegative integers such that*

(5) $$\sum_{i=1}^{t} (D_i + 1) \geq \Delta(G) + 1,$$

then there is a partition $(V_1, \ldots, V_t)$ *of* $V(G)$ *s.t.*

$$\Delta(G[V_i]) \le D_i \qquad \forall i \in [t].$$

Proof. Choose a partition $(V_1, \ldots, V_t)$ of $V(G)$ to minimize $\sum_{i=1}^{t} |E(G[V_i])|/D_i$. If $v \in V_i$ and $d_{G[V_i]}(v) \ge D_i + 1$, then there is $j \in [t]$ s.t. $d_{G[V_j \cup \{v\}]}(v) \le D_j$. Move v there. $\square$

Conjecture (Correa-Havet-Sereni, 2009). *There exists an integer* $k_0 \ge 3$ *such that for each* $k \ge k_0$*, the vertex set of every planar graph* G *with maximum degree at most* $2k+2$ *can be partitioned intosubsets* V_1 *and* V_2 *such that* $\Delta(G[V_i]) \le k$ *for* $i = 1, 2$.

Theorem 24 (Stiebitz 1996 (was Thomassen's Conjecture), Th. 6.2.30 in the book)**.** *If* $\delta(G) \ge s+t+1$*, then there is a partition* (A, B) *of* $V(G)$ *s.t.* $\delta(G[A]) \ge s$ *and* $\delta(G[B]) \ge t$.

Proof. An (s,t)-triple of G is a partition (A, B, C) of $V(G)$ s.t. $\delta(G[A]) \ge s$ and $\delta(G[B]) \ge t$. We want to prove that G has an (s,t)-triple (A, B, C) with $C = \emptyset$.

Choose a minimum $A' \subset V(G)$ with $\delta(G[A']) \ge s$. By minimality, $G[A']$ is s-degenerate. If $G[\overline{A'}]$ is not t-degenerate, then there is $B'' \subseteq \overline{A'}$ with $\delta(G[B']) \ge t+1$. In the latter case, G has an (s,t)-triple $(A', B', \overline{A' \cup B'})$. In the former case, among partitions (A', B') of $V(G)$ into an s-degenerate and a t-degenerate induced subgraphs, choose (A, B) maximizing

$$f(A,B) = |E(G[A])| + |E(G[B])| + t|A| + s|B|.$$

If there is $v \in A$ with $d_{G[A]}(v) \le s-1$, then by moving v into B we decrease $|E(G[A])|$ by at most $s-1$, increase $|E(G[B])|$ by at least $t+2$, decrease $t|A|$ by t and increase $s|B|$, the net change for $f(A,B)$ positive, a contradiction. Thus $\delta(G[A]) \ge s$. Similarly, $\delta(G[B]) \ge t$, proving the theorem.

So, in any case G has an (s,t)-triple $(A', B', \overline{A' \cup B'})$. Among such triples choose one with maximum $|A' \cup B'|$. If there is $x \in V(G) - A' - B'$, then by maximality, v has at most $t-1$ neighbors in B' and hence at least $s+1$ neighbors in $\overline{B'}$. It follows that each $u \in \overline{B'}$ has at least s neighbors in $\overline{B'}$. So we have an (s,t)-triple $(\overline{B'}, B', \emptyset)$. $\square$

———————————— **Here Lecture 11 ended.**